Curve Criterion

A function f of several variables satisfies $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$ iff for all continuous space curves $\vec{r}(t)$ with $\lim_{t \to \infty} \vec{r}(t) = a$ and $\vec{r}(t) \neq a$ for all t

we have $\lim_{t \to \infty} f(\vec{r}(t)) = L$

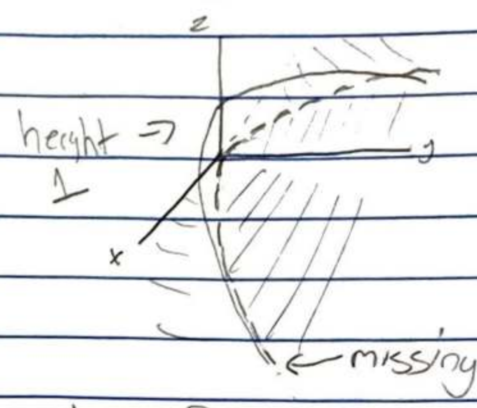
Idea: Find two curves $\vec{r}_0(t)$ and $\vec{r}_1(t)$ with $\lim_{t \to \infty} \vec{r}_i(t) = \vec{a}$ and $\lim_{t \to \infty} f(\vec{r}_0(t)) \neq \lim_{t \to \infty} f(\vec{r}_1(t))$

Note! The collection of lines given by $l_{a,b}(t) = \vec{a} + t\langle a,b \rangle$ a good collection of curves to start b/c

$$\lim_{t \to 0} l_{a,b}(t) = \vec{a}$$

This way isn't always good enough

Ex: Let $f(x,y) = \begin{cases} 1 & \text{if } y = x^2 \\ 0 & \text{otherwise} \end{cases}$



Claim: $\lim_{\vec{x} \to \vec{0}} f(\vec{x})$ does not exist

For lines $l_{a,b}(t)$: $\lim_{t \to 0^+} f(l_{a,b}(t)) = \lim_{t \to 0^+} f(at, bt)$

Off of the origin (at, bt) satisfies $bt = (at)^2$ at most once (solve for t)

$\therefore f(at, bt) = 0$ for all but finitely many times

$\therefore \lim_{t \to 0^+} f(l_{a,b}(t)) = 0$

On the other hand,

Letting $\vec{r}(t) = \langle t, t^2 \rangle$, we see $f(\vec{r}(t)) = f(t, t^2) = 1$ for all t

$\therefore \lim_{t \to 0} f(\vec{r}(t)) = \lim_{t \to 0} 1 = 1$

Thus $\lim_{\vec{x} \to \vec{0}} f(\vec{x})$ does not exist $\blacksquare$

How can we show limits do exist?

A trick: Try polar coordinates

Ex: Does $\lim_{(x,y)\to 0,0} \frac{\sin(x^2+y^2)}{x^2+y^2}$ exist?

Sol: Convert limit to polar coordinates: $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$

$(x,y) \to 0,0$ iff $\vec{r} \to 0^+$

$$\lim_{(x,y)\to 0,0} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{\vec{r}\to 0^+} \frac{\sin((r\cos\theta)^2 + (r\sin\theta)^2)}{(r\cos\theta)^2 + (r\sin\theta)^2}$$

if it exist

$$= \lim_{\vec{r}\to 0^+} \frac{\sin(r^2(\cos^2\theta + \sin^2\theta))}{r^2(\cos^2\theta + \sin^2\theta)}$$

$$= \lim_{\vec{r}\to 0^+} \frac{\sin(r^2)}{r^2} \Rightarrow \frac{0}{0} \text{ type}$$

$$\overset{\boxed{LH}}{=} \lim_{r\to 0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0^+} \cos(r^2) = \cos(0^2) = 1$$

Ex Does $\lim_{(x,y)\to 0,0} \frac{x^2-y^2}{x^2+y^2}$ exist?

Sol $$\lim_{(x,y)\to 0,0} \frac{x^2-y^2}{x^2+y^2} = \lim_{\vec{r}\to 0^+} \frac{(r\cos\theta)^2 - (r\sin\theta)^2}{(r\cos\theta)^2 + (r\cos\theta)^2}$$

$$= \lim_{\vec{r}\to 0^+} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2(\cos^2\theta + \sin^2\theta)} = \lim_{\vec{r}\to 0^+} \cos 2\theta = \cos(2\theta)$$

Depends on θ

If we approach along angle $\theta = \pi/2$, we expect $\lim_{\vec{x}\to 0} f(\vec{x}) = \cos(2(\pi/2)) = -1$

If $\theta = 0$ $\cos(0) = 1$ $\therefore \lim_{\vec{x}\to 0} \frac{x^2-y^2}{x^2+y^2}$ DNE ☒

Def: A function f of n variables is continuous at $\vec{a} \in dom(f)$ when

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$$

f is continuous on set D when f is cts. at very member of D

Ex Every polynomial in n-variables is cts. on $\mathbb{R}^n$

Ex Every rational function of n-variables is cts, on its domain

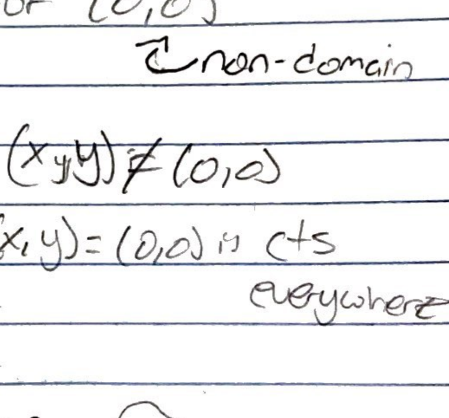
↳ $\frac{x^2-y^2}{x^2+y^2}$ is cts on its domain (everywhere but $(0,0)$)

Ex $\frac{\sin(x^2+y^2)}{x^2+y^2}$ is cts. everywhere but $(0,0)$

non-domain

OTOH $f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ is cts everywhere

Derivatives of Multivariable Functions



Idea: The derivative measures change in output from corresponding small change in input IN SOME DIRECTION

Def: Let f be a function of n variables and take $\vec{u}$ a unit vector in $\mathbb{R}^n$

Let $\vec{a} \in dom(f)$

The directional derivative of f at $\vec{a}$ in direction

of $\vec{u}$ is $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \dfrac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$

Ex Compute the directional derivative of $f(x,y) = xy$ at $\vec{a} = \langle 1, 3 \rangle$ in direction $\vec{u} = \frac{1}{2}\langle \sqrt{2}, \sqrt{2} \rangle$

Sol: $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \dfrac{f(\vec{a} + h\vec{u}) - f(\vec{a})}{h}$

$= \lim_{h \to 0^+} \dfrac{f(1 + \frac{\sqrt{2}}{2}h, 3 + \frac{\sqrt{2}}{2}h) - f(1,3)}{h}$

$= \lim_{h \to 0^+} \dfrac{(1 + \frac{\sqrt{2}}{2}h)(3 + \frac{\sqrt{2}}{2}h) - 1 \cdot 3}{h}$

$= \lim_{h \to 0^+} \dfrac{3 + h\left(\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) + h^2 - 3}{h}$

$= \lim_{h \to 0^+} \dfrac{h(2\sqrt{2} + h)}{h} = \lim_{h \to 0^+} (2\sqrt{2} + h) = 2\sqrt{2} + 0 = 2\sqrt{2}$ $\boxtimes$

Exercise: Repeat exercise with $\vec{a} = \langle x, y \rangle$

Note: The directional derivative is very general

Def: Let f be a function of n-variable and let $\vec{e}_k$ be the standard k-th basis vector in $\mathbb{R}^n$, i.e.
$\vec{e}_k = \langle 0, 0, \ldots, 1, \ldots, 0 \rangle$
↑ k-th position
The k-th partial derivative of f
$D_{\vec{e}_k} f(\vec{a})$